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# PROOF OF LÁSZLÓ FEJES TÓTH'S ZONE CONJECTURE

Zilin Jiang · Alexandr Polyanskii

**Abstract.** A zone of width $\omega$ on the unit sphere is the set of points within spherical distance $\omega/2$ of a given great circle. We show that the total width of any collection of zones covering the unit sphere is at least $\pi$, answering a question of Fejes Tóth from 1973.

## 1 Introduction

A *plank* (or *slab*, or *strip*) of width $w$ is a part of the $d$-dimensional Euclidean space $\mathbb{R}^d$ that lies between two parallel hyperplanes at distance $w$. Given a convex body $C$, its *width* is the smallest $w$ such that a plank of width $w$ covers $C$. In 1932, in the context of "degree of equivalence of polygons", Tarski wrote in [Tar32],[1] and we quote from its English translation [MMS14, Chapter 7.4],

> The width of the narrowest strip covering a plane figure $F$ we shall call the *width of figure* $F$, and we shall denote it by the symbol $\omega(F)$....
>
> **A.** *If a figure $F$ contains in itself, as a part, a disk with diameter equal to the width of the figure (for example, if figure $F$ is a disk or a parallelogram) and if, moreover, we subdivide this figure into any $n$ parts $C_1, C_2, \ldots, C_n$, then*
>
> $$\omega(F) \leq \omega(C_1) + \omega(C_2) + \cdots + \omega(C_n).$$
>
> *Proof.* For the case in which figure $F$ is a disk, the proof is an almost word-for-word repetition of the proof of lemma I in the cited article by Moese. For the general case,...

Here "the cited article by Moese" refers to [Moe32][1]. For a long period of time, the authors of the current paper mistakenly overlooked the contribution of Moese.

The following conjecture, which is attributed to Tarski, seems to first appear in [Ban50].

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[1] See [MMS14, Chapter 7.3, 7.4] for English translations of [Moe32] and [Tar32].

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**Tarski's plank problem.** If a convex body of width $w$ is covered by a collection of planks in $\mathbb{R}^d$, then the total width of the planks is at least $w$.

It took almost twenty years before Bang proved Tarski's conjecture in his memorable papers [Ban50, Ban51]. At the end of his paper, Bang asked whether his theorem could be strengthened by asking that the width of each plank should be measured relative to the width of the convex body being covered, in the direction normal to the plank. Although Bang's conjecture still remains open, Ball [Bal91] established it for centrally symmetric convex bodies (see [Bal01] for the complex analogue). Ball also used Bang's proof to improve the optimal density of lattice packings [Bal92].

Variants of Tarski's plank problem continue to generate interest in the geometric and analytic aspects of coverings of a convex body (see [Bez13] for a recent survey). Most of these variants find its generalization in spherical geometry. Bezdek and Schneider [BS10] showed that the total inradius of the spherical convex domains[2] covering the $n$-dimensional unit sphere is at least $\pi$ (see [AK12, Section 6] for a simplified proof by Akopyan and Karasev, and see [Kad05] for the Euclidean version by Kadets).

In this paper, we are concerned with a spherical analogue of a plank: a *zone* of width $\omega$ on the 2-dimensional unit sphere is defined as the set of points within spherical distance $\omega/2$ of a given great circle. In 1973, Fejes Tóth [Tót73] conjectured that if $n$ equal zones cover the sphere then their width is at least $\omega_n = \pi/n$. Rosta [Ros72] and Linhart [Lin74] proved the special case of 3 and 4 zones respectively. Lower bounds for $\omega_n$ were established by Fodor, Vígh and Zarnócz [FVZ16].

Fejes Tóth also formulated the generalized conjecture, which has been reiterated in [BMP05, Chapter 3.4, Conjecture 5] and the arXiv version of [AK12, Conjecture 8.3], for any set of zones (not necessarily of the same width) covering the unit sphere.

**Fejes Tóth's zone conjecture.** The total width of any set of zones covering the sphere is at least $\pi$.

We completely resolve this conjecture and generalize it for the $d$-dimensional unit sphere $S^d$. Hereafter, all $d$-spheres are embedded in $\mathbb{R}^{d+1}$ and centered at the origin. For us, a *great sphere* is the intersection of a $d$-sphere and a hyperplane passing through the origin. This extends the notion of a great circle on a 2-dimensional sphere. Given a great sphere $C$, a *zone* $P$ of width $\omega$ on $S^d$ is defined as the set of points within spherical distance $\omega/2$ of $C$, and the *central hyperplane* of $P$ is the hyperplane through $C$.

**Theorem 1.** *The total width of any collection of zones $P_1, \dots, P_n$ covering the unit $d$-sphere is at least $\pi$. For all $i \in [n]$, let $2\alpha_i$ be the width of $P_i$, and let $\ell_i$ be the line through the origin perpendicular to the central hyperplane of $P_i$. Equality holds if and only if after reordering the zones, $\ell_1, \dots, \ell_n$ are coplanar lines in clockwise order*

[2] A *convex spherical domain* is a domain any two points of which can be joined by an arc of a great circle lying in the domain. The inradius of the domain $C$ is the spherical radius of the largest cap contained in $C$.